SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2012
YEAR 11 Mathematics Extension
HSC Task #1

Mathematics Extension

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answers must be given in simplest exact form unless otherwise stated.

Total marks - 77

Section I (10 marks)
- Answer Questions 1-10 on the Multiple Choice answer sheet provided.

Section II (67 marks)

- For Questions 11-13, start a new answer booklet for each question.

Examiner: *D.McQuillan*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Section I (10 marks)

Use the multiple-choice answer sheet for Questions 1-10

1 What is the acute angle to the nearest degree between the lines $y = 7 - 4x$ and $2x - 3y - 6 = 0$?

(A) $26°$

(B) $42°$

(C) $70°$

(D) $75°$

2 When polynomial $P(x)$ is divided by $x^2 + x - 6$ the remainder is $7x + 13$. What is the remainder when $P(x)$ is divided by $x + 3$?

(A) -8

(B) -5

(C) 34

(D) 55

3 What is the exact value of $\cos 165°$?

(A) $\frac{-\sqrt{2}-\sqrt{6}}{4}$

(B) $\frac{-\sqrt{2}+\sqrt{6}}{4}$

(C) $\frac{-\sqrt{6}-\sqrt{2}}{4}$

(D) $\frac{-\sqrt{6}+\sqrt{2}}{4}$

4 What are the coordinates of the point P that divides internally the interval joining the points A(1, 2) and B(7, 5) in the ratio 2:1 ?

(A) (3, 3)

(B) (3, 4)

(C) (5, 3)

(D) (5, 4)

5 What is the solution to the inequality $\frac{3}{x-2} \geq 4$?

(A) $x < -2$ and $x \geq \frac{11}{4}$

(B) $x > -2$ and $x \leq -\frac{11}{4}$

(C) $x < 2$ and $x \geq \frac{11}{4}$

(D) $x > 2$ and $x \leq \frac{11}{4}$

6 The average rate of change of the function $f(x) = x^3 - \sqrt{x+1}$ between $x = 0$ and $x = 3$ is

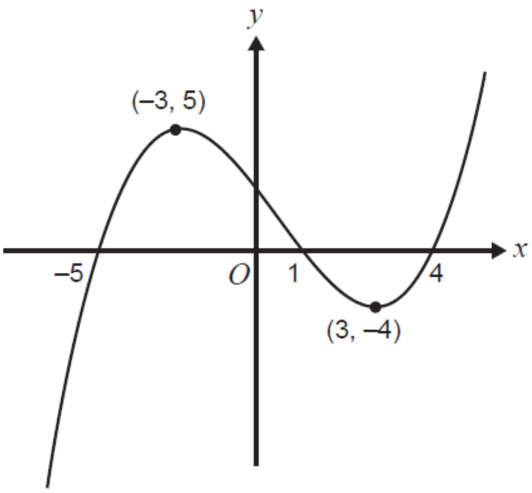
(A) 12

(B) $\frac{26}{3}$

(C) $\frac{25}{3}$

(D) 8

7



For the graph of $y = f(x)$ shown above $f'(x)$ is negative when

(A) $-3 < x < 3$

(B) $-3 \leq x \leq 3$

(C) $x < -3$ or $x > 3$

(D) $x < 5$ or $1 < x < 4$

8 Let $f'(x) = g'(x) + 3$, $f(0) = 2$ and $g(0) = 1$.

Then $f(x)$ is given by

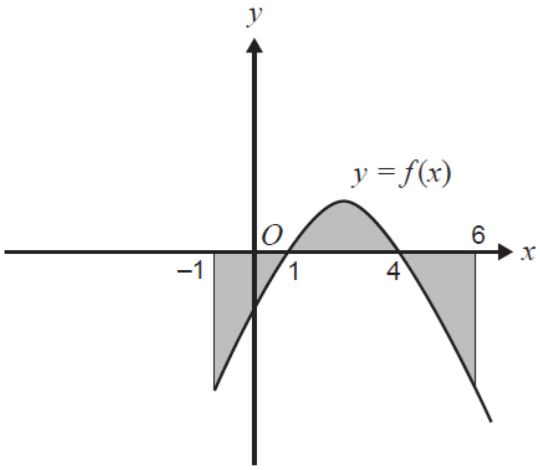
(A) $f(x) = g(x) + 3x + 1$

(B) $f(x) = g(x) + 3x$

(C) $f(x) = 1$

(D) $f(x) = g'(x) + 3$

9



The total area of the shaded regions in the diagram is given by

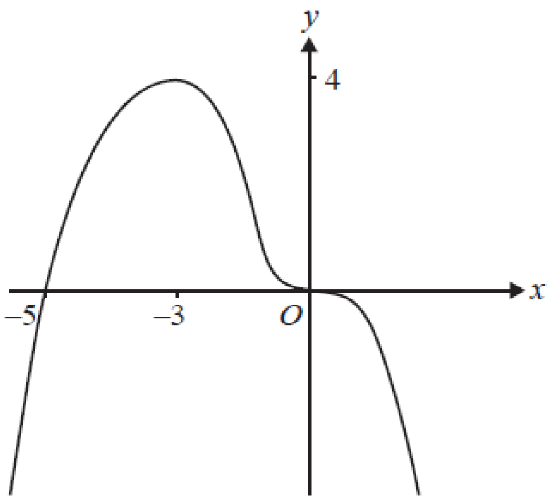
(A) $\int_{-1}^{6} f(x)\,dx$

(B) $\int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx + \int_{4}^{6} f(x)\,dx$

(C) $-\int_{1}^{-1} f(x)\,dx + \int_{1}^{4} f(x)\,dx - \int_{6}^{4} f(x)\,dx$

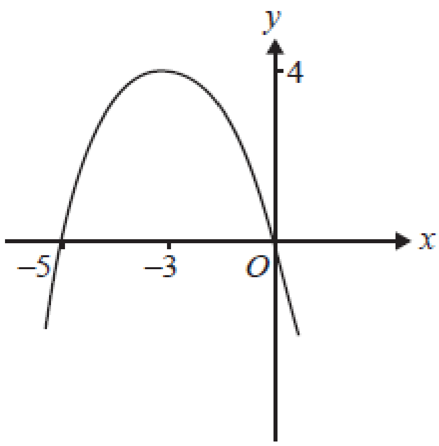
(D) $\int_{1}^{-1} f(x)\,dx + \int_{1}^{4} f(x)\,dx + \int_{6}^{4} f(x)\,dx$

10 The graph of $y = f(x)$ is show below.

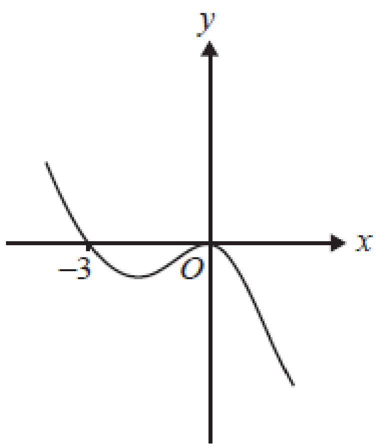


Which of the following could be the graph of the derivative function $y = f'(x)$?

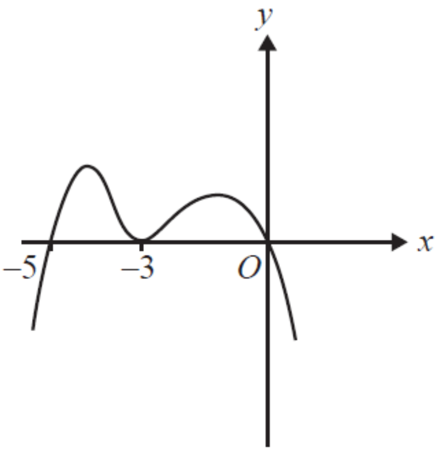
(A)



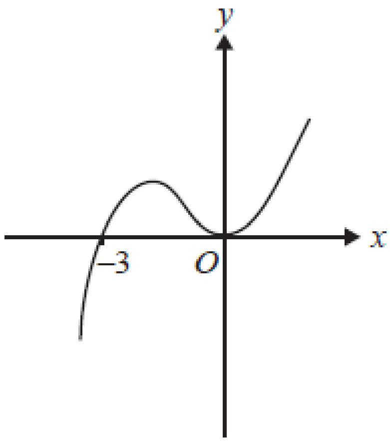
(B)



(C)



(D)



Section II

Start a new writing booklet for each Question in section II

Question 11 (21 marks)

(a) Find **[4]**

(i)

$$\int (5 - 2x + 6x^2)dx$$

(ii)

$$\int \sqrt[3]{x}\, dx$$

(iii)

$$\int \frac{5x^4 + x - 4}{2x^6} dx$$

(b) Solve **[4]**

(i)

$$\frac{4}{|x-1|} > 1$$

(ii)

$$\frac{4}{(x-1)^2} > 1$$

(c) Given $P(x) = x^3 - x^2 - 3x - 1$ **[4]**

(i) Evaluate $P(-1)$.

(ii) Hence solve $P(x) = 0$.

(d) The roots of $x^3 - 4x^2 - 9 = 0$ are α, β, γ. Find the value of **[4]**

(i) $(\alpha + \beta + \gamma) - \alpha\beta\gamma$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(e) Prove that **[2]**

$$\frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A$$

(f) Sketch the graph of a function that satisfies all of the given conditions. **[3]**

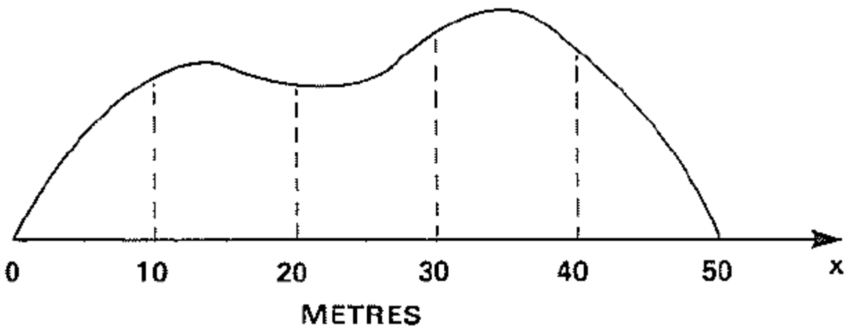
- $f'(5) = 0$
- $\lim_{x \to 3} f(x) = -\infty$
- $f'(x) < 0$ for $x < 3$ and for $x > 5$
- $f'(x) > 0$ for $3 < x < 5$

End of Question 11

Start a new writing booklet for each Question in section II

Question 12 (23 marks)

(a) During a survey the area of an irregular headland was to be found. The surveyor used a base line divided into five equal sub-intervals, each of width ten metres. Offset measurements $f(x)$ were taken at each value x across the base line and tabulated as below **[3]**



x (metres)	0	10	20	30	40	50
Offset (metres)	0	13.2	13.8	17	17.4	0

Use the trapezoidal rule to find the approximate area of the headland.

(b) **[4]**

(i) Rewrite $2\cos x + \sqrt{5}\sin x$ in the form $Rcos(x - \alpha)$.

(ii) Hence solve $2\cos x + \sqrt{5}\sin x = 2$ for $0° \leq x < 360°$, to the nearest degree.

(c) Sketch $y = x^4 - 6x^2 - 8x + 1$ showing all stationary points. **[5]**

(d) If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be? **[2]**

(e) Find the general solution of $\cos 2x = \frac{1}{2}$ (answer in terms of radians). **[2]**

(f) **[4]**

(i) Show that

$$\frac{d}{dx}\left(\frac{x}{a^2\sqrt{a^2 - x^2}}\right) = \frac{1}{\sqrt{(a^2 - x^2)^3}}$$

(ii) Hence evaluate

$$\int_0^2 \frac{dx}{\sqrt{(9 - x^2)^3}}$$

(g) Prove that the polynomial

$$P(x) = x^{101} + x^{51} + x + 1$$

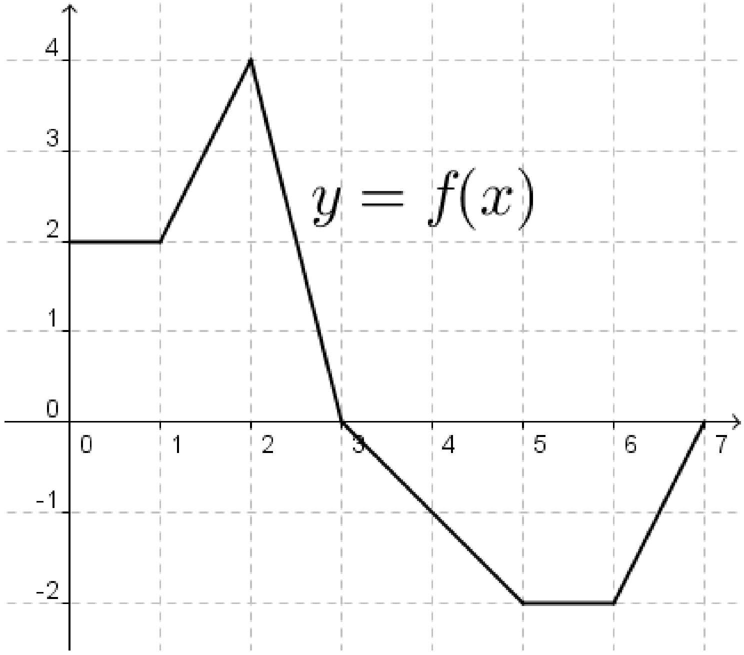
has neither a local maximum nor a local minimum. **[3]**

End of Question 12

Start a new writing booklet for each Question in section II

Question 13 (23marks)

(a) Let $g(x) = \int_0^x f(t)dt$ where $f(x)$ is the function whose graph is shown. **[6]**



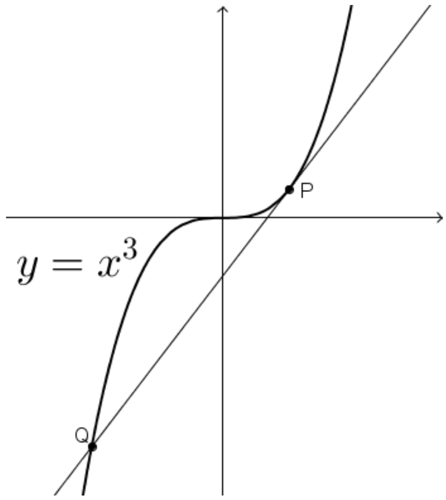
(i) Evaluate $g(1)$, $g(3)$ and $g(6)$.

(ii) On what intervals is $g(x)$ increasing?

(iii)Where does $g(x)$ have a maximum value?

(iv)Sketch the graph of $y = g(x)$.

(b) Let P be a point on the curve $y = x^3$ and suppose the tangent line at P intersects the curve again at Q. Prove that the slope at Q is four times the slope at P. **[4]**



(c) A square room with length and breadth 4 metres is 2.4 metres high. Calculate, to the nearest minute, the angle which a string line will make with the floor if it is stretched from the centre of the ceiling to one corner of the floor. **[3]**

(d) Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle. **[4]**

(e) If $4\sin 2x + 2\cos 2x = \tan x$ **[6]**

(i) Prove that $\tan x$ is a root of the equation $t^3 + 2t^2 - 7t - 2 = 0$.

(ii) Verify that $t = 2$ is one root and find the exact values of the others.

(iii) Hence solve, $4\sin 2x + 2\cos 2x = \tan x$ for $0° < x < 360°$, to the nearest degree.

End of Question 13

End of Exam

Student Number: SOLNS

Mathematics Extension
Task 1 2012

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✖ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

A ✖ B ✖ (correct) C ○ D ○

Section I: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. C — C
2. A — A
3. A, C — A or C
4. D — D
5. D — D
6. B — B
7. A — A
8. A — A
9. D — D
10. B — B

Q11 (EXT1)

(a) (i) $\int (5-2x+6x^2)\,dx = 5x - x^2 + 2x^3 + c.$ [1]

(ii) $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx.$

$= \dfrac{x^{4/3}}{4/3} + c.$ [1]

$= \dfrac{3x^{4/3}}{4} + c$ OR $\dfrac{3x\sqrt[3]{x}}{4} + c.$

(iii) $\int \dfrac{5x^4 + x - 4}{2x^6}\,dx = \int \left(\dfrac{5}{2}x^{-2} + \dfrac{x^{-5}}{2} - 2x^{-6}\right)dx$

$= \dfrac{5x^{-1}}{-2} + \dfrac{x^{-4}}{-8} - \dfrac{2x^{-5}}{-5} + c$

$= -\dfrac{5}{2}x^{-1} - \dfrac{1}{8}x^{-4} + \dfrac{2}{5}x^{-5} + c$ [2]

OR $= \dfrac{-5}{2x} - \dfrac{1}{8x^4} + \dfrac{2}{5x^5} + c$

OR $\dfrac{-100x^4 - 5x + 16}{40x^5} + c$

(b) (i) $\dfrac{4}{|x-1|} > 1$

$\Rightarrow |x-1| < 4$

ie $-4 < x-1 < 4$

ie $-3 < x < 5$, $x \neq 1$

[2]

(ii) $\dfrac{4}{(x-1)^2} > 1$

$4 > (x-1)^2.$

$-2 < x-1 < 2$

$-1 < x < 3,\ x \neq 1$

[2]

(c) $P(x) = x^3 - x^2 - 3x - 1$.

(i) $P(-1) = -1 - 1 + 3 - 1$

$= 0$.

$\therefore$ $x+1$ is a factor.

	1	−1	−3	−1
−1		−1	2	1
	1	−2	−1	0

[using Synthetic or Long division]

$\therefore$ $x^2 - 2x - 1$ is the other factor.

$\therefore$ $P(x) = (x+1)(x^2-2x-1)$

4

If $P(x) = 0$

$(x+1)(x^2-2x-1) = 0$

$x = -1, \frac{2 \pm \sqrt{8}}{2}$

$x = -1; 1 \pm \sqrt{2}$.

(d) given $x^3 - 4x^2 - 9 = 0$ with roots α, β & γ.

$\alpha + \beta + \gamma = 4$

$\alpha\beta + \alpha\gamma + \beta\gamma = 0$

$\alpha\beta\gamma = 9$.

$\therefore$ (i) $(\alpha+\beta+\gamma) - \alpha\beta\gamma = 4 - 9$

$= -5$

4

(ii) $(\alpha+2)(\beta+2)(\gamma+2) = \alpha\beta\gamma + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$+ 4(\alpha+\beta+\gamma) + 8$.

$= 9 + 0 + 16 + 8$

$= 33$

(e) Prove that $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.

$$LHS = \frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A-1)}$$

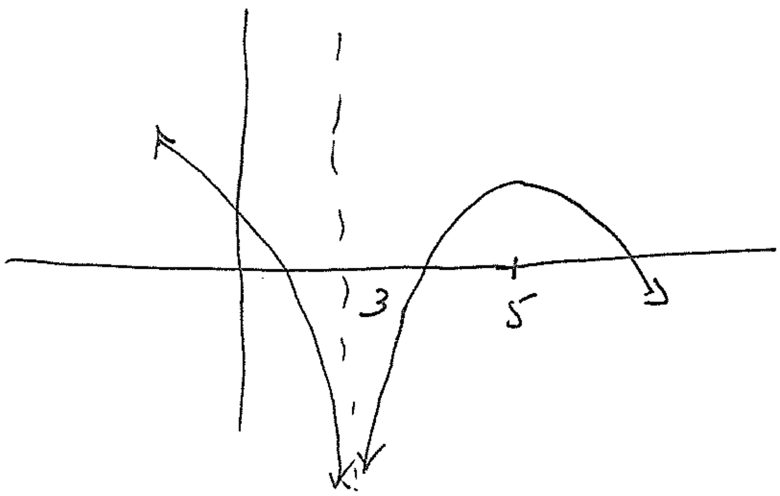
2

$$= \frac{2\sin^2 A}{2\cos^2 A}$$

$$= \tan^2 A$$

$$= RHS.$$

(f)



3

Ext 1 Ass 1 2012

Question 12

(a) $A \doteq \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 ...)]$

$= 5[0 + 0 + 2(13.2 + 13.8 + 17 + 17.4)]$

$= 614 \text{ m}^2$ [3]

(b) (i) $2\cos x + \sqrt{5}\sin x = R\cos(x - a)$

$R = \sqrt{2^2 + \sqrt{5}^2}$

$= 3$

$\therefore \text{LHS} = 3\left(\frac{2}{3}\cos x + \frac{\sqrt{5}}{3}\sin x\right)$

$\therefore \sin a = \sqrt{5}/3$

$a \doteq 48°11'23''$

$\doteq 0.8411^c$ [2]

(ii) Exp $= 3\cos(x - 0.8411)$

$= 3\cos(x - 48°11'23'')$

$3\cos(x - 48°11'23'') = 2$

$x - 48°11'23'' = \cos^{-1}(2/3)$

$= -48°11'23'', 48°11'23''$

$x = 0°$ or $96°22'46''$

$\doteq 0°$ or $96°$ [2]

(c) $y = x^4 - 6x^2 - 8x + 1$

$y' = 4x^3 - 12x - 8$

$y'' = 12x^2 - 12$

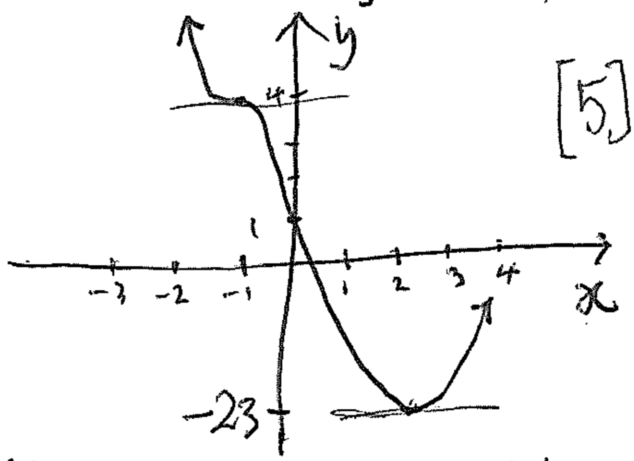
$y' = 0$ when

$4x^3 - 12x - 8 = 0$

$x^3 - 3x - 2 = 0$

$x = -1, -1, 2$

$y = 4, -23$



[5]

(d) Min of $f(4)$ will be when $f'(x) = 2$ (least value

$\therefore$ Let $m = 2$, $(1, 10)$

$y - 10 = 2(x - 1)$

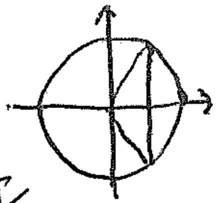
$y = 2x + 8$

When $x = 4$ $y = 8 + 8$

$= 16$

$\therefore f(4) \geq 16$ [2]

(e) $\cos 2x = \frac{1}{2}$



$2x = 2n\pi \pm \frac{\pi}{3}$

$x = n\pi \pm \frac{\pi}{6}$ [2]

$n \in \mathbb{Z}$

(f) (i) $\frac{d}{dx}\left(\frac{x}{a^2\sqrt{a^2-x^2}}\right)$

$= \frac{1}{a^2}\left[\frac{\sqrt{a^2-x^2}\cdot 1 - x\cdot\frac{1}{2\sqrt{a^2-x^2}}\cdot(-2x)}{(a^2-x^2)}\right]$

$= \frac{1}{a^2}\left[\frac{(a^2-x^2)+x^2}{\sqrt{(a^2-x^2)^3}}\right]$

$= \frac{1}{\sqrt{(a^2-x^2)^3}}$ [2]

as required.

(ii) $\int_0^2 \frac{dx}{\sqrt{(9-x^2)^3}}$

$= \left[\frac{1}{9}\frac{x}{\sqrt{9-x^2}}\right]_0^2$

$= \frac{1}{9}\left[\frac{2}{\sqrt{5}} - 0\right]$

$= \frac{2}{9\sqrt{5}}$ [2]

(g) $P(x) = x^{101} + x^{51} + x + 1$

$P'(x) = 101x^{100} + 51x^{50} + 1$

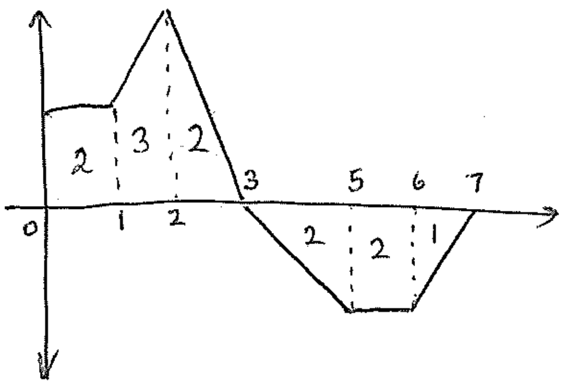
$> 0 \quad \forall x$

$\therefore$ P(x) is monotonic increasing.

$\therefore$ P(x) has no turning points. [3]

Question 13

a)



i) $g(1) = 2$

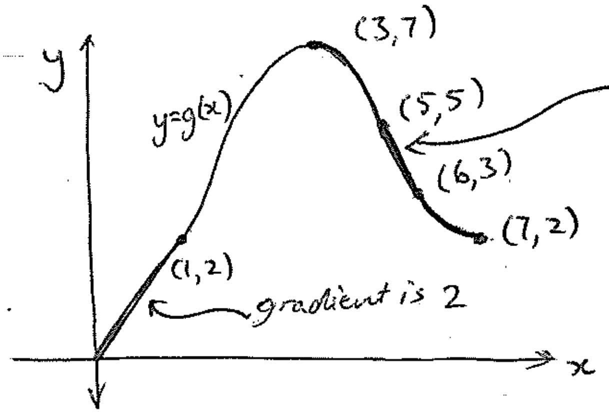
$g(3) = 7$

$g(6) = 3$

ii) $0 < x < 3$

iii) when $x = 3$

iv)



b) let P have coordinates (x_1, y_1)

Q have coordinates (x_2, y_2)

$y = x^3$

$y' = 3x^2$

at P

$m_T = 3x_1^2$

equation of tangent at P: $y - y_1 = 3x_1^2(x - x_1)$

$y = 3x_1^2 x - 3x_1^3 + y_1$

$y = 3x_1^2 x - 2x_1^3$ ——① since $y_1 = x_1^3$

$y = x^3$ ——————②

sub ① into ②

$x^3 = 3x_1^2 x - 2x_1^3$

$x^3 - 3x_1^2 x + 2x_1^3 = 0$

let roots be x_1, x_1, x_2 (since tangent at $P(x_1, y_1)$)

$x_1 + x_1 + x_2 = 0$

$2x_1 + x_2 = 0$

$x_2 = -2x_1$

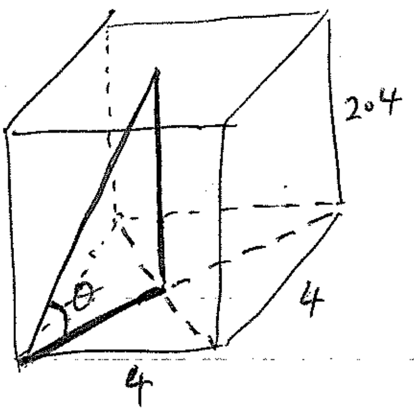
gradient of tangent at $Q = 3(-2x_1)^2$

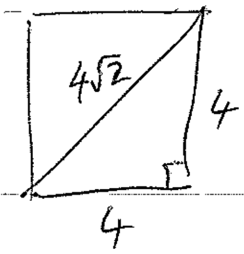
$= 12x_1^2$

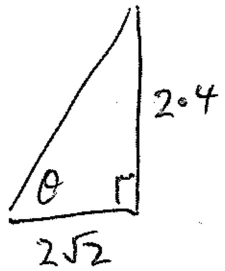
$= 4(3x_1^2)$

= four times gradient of tangent at P

c)



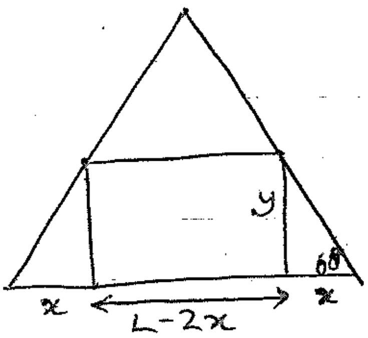




$\tan\theta = \frac{2 \cdot 4}{2\sqrt{2}}$

$\theta = 40°19'$

d)



$A = (L-2x)y$

$\tan 60° = \frac{y}{x}$

$\sqrt{3} = \frac{y}{x}$

$y = \sqrt{3}x$

$A = (L-2x)\sqrt{3}x$

$A = \sqrt{3}Lx - 2\sqrt{3}x^2$

$A' = \sqrt{3}L - 4\sqrt{3}x$

$A'' = -4\sqrt{3}$ ⌒

let $A' = 0$

$\sqrt{3}L - 4\sqrt{3}x = 0$

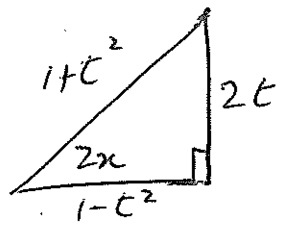
$4\sqrt{3}x = \sqrt{3}L$

$x = \frac{L}{4}$

since $A'' < 0$, Maximum area when $x = \frac{L}{4}$

Dimensions are $\frac{L}{2} \times \frac{\sqrt{3}L}{4}$

e) i) let $t = \tan x$



$$4\sin 2x + 2\cos 2x = \tan x$$
$$4\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) = t$$
$$8t + 2 - 2t^2 = t + t^3$$
$$t^3 + 2t^2 - 7t - 2 = 0$$

ii) let $P(t) = t^3 + 2t^2 - 7t - 2$

$$P(2) = (2)^3 + 2(2)^2 - 7(2) - 2$$
$$= 0$$

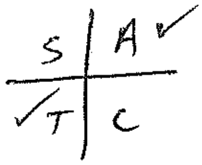
$\therefore t = 2$ is a root of $P(t) = 0$.

$$
\begin{array}{r|l}
 & t^2 + 4t + 1 \\
t - 2 & t^3 + 2t^2 - 7t - 2 \\
 & t^3 - 2t^2 \\
 & 4t^2 - 7t \\
 & 4t^2 - 8t \\
 & t - 2 \\
 & t - 2 \\
 & 0
\end{array}
$$

$$\therefore P(t) = (t-2)(t^2 + 4t + 1)$$
$$t^2 + 4t + 1 = 0$$
$$t^2 + 4t + 4 = -1 + 4$$
$$(t+2)^2 = 3$$
$$t + 2 = \pm\sqrt{3}$$

$t = -2 \pm \sqrt{3}$ are the other roots

iii) $\tan x = 2$

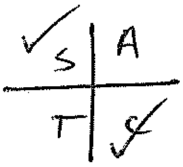


$\tan\alpha = 2$

$\alpha \doteqdot 63^\circ$

$x = 63^\circ, 243^\circ$

$\tan x = -2+\sqrt{3}$

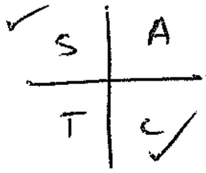


$\tan\alpha = 2-\sqrt{3}$

$\alpha = 15^\circ$

$x = 165^\circ, 345^\circ$

$\tan x = -2-\sqrt{3}$



$\tan\alpha = 2+\sqrt{3}$

$\alpha = 75^\circ$

$x = 105^\circ, 285^\circ$

$\therefore x = 63^\circ, 105^\circ, 165^\circ, 243^\circ, 285^\circ, 345^\circ$